According to the latter (see Ref. 6), the operator corresponding to the classical generalized momentum operator p_q is the Hermitian part of the operator $p_q = -i\hbar\, \partial/\partial q$, that is, $p_q{}^H$. This is identical with the operator obtained from Weyl, Born–Jordan, or symmetrized ordering.

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Comments on the Correspondence Principles of Quantum Mechanical Operators

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In an article by Margenau and Cohen various correspondence principles were described in connection with Weyl, Born–Jordan, and symmetrized ordering of quantum mechanical operators. In this article we make an interesting comparison between the aforementioned ordering process and our previous prescriptions.

1. INTRODUCTION

It will be shown that in generalized coordinates Weyl, Born–Jordan, and symmetrized ordering[1] of operators and our previous prescriptions[2,3] for ordering operators in generalized coordinates have interesting connections. In this article we present a comparison between our previous prescriptions and the three mentioned correspondence rules applied to two fundamental operators in quantum mechanics: the generalized canonical momenta p_q and the function $p_q{}^2F$, where F is a function of generalized coordinates.

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2. THE OPERATOR CORRESPONDING TO THE CLASSICAL QUANTITY p_q^2F: WEYL CORRESPONDENCE

If we have a function of generalized momenta and coordinates such as p_q^2F, then in order to find the corresponding quantum mechanical operator, we must first transform the classical quantity p_q^2F into Cartesian coordinates. The function p_q^2F then becomes[4]

$$p_q^2F = p_x^2(\partial x/\partial q)^2 F$$

where p_x is the Cartesian momentum, x is the Cartesian coordinate, and q is the generalized coordinate.[1] We now must transform the quantity $p_x^2(\partial x/\partial q)^2 F$ into its related quantum operator by substituting $p_x \to -i\hbar\partial/\partial x$ and order the operator $p_x^2(\partial x/\partial q)^2 F$ according to the Weyl prescription.[5] After doing this we arrive at the operator $(p_x^2(\partial x/\partial q)^2 F)^W$, the Weyl ordered operator, given as

$$\left(p_x^2\left(\frac{\partial x}{\partial q}\right)^2 F\right)^W = \left(\frac{\partial x}{\partial q}\right)^2 F p_x^2 - i\hbar\left[\frac{\partial F}{\partial x}\left(\frac{\partial x}{\partial q}\right)^2 + 2F\frac{\partial x}{\partial q}\frac{\partial(\partial x/\partial q)}{\partial x}\right]p_x + \hbar^2 G(x) \quad (1)$$

where G is a function of x.

Transforming back to generalized coordinates, it is not difficult to see that

$$\left(p_x^2\left(\frac{\partial x}{\partial q}\right)^2 F\right)^W = -\hbar^2 F\frac{\partial^2}{\partial q^2} - \hbar^2 F\frac{1}{g}\frac{\partial g}{\partial q}\frac{\partial}{\partial q} - \hbar^2\frac{\partial F}{\partial q}\frac{\partial}{\partial q} + \hbar^2 G(q) \quad (2)$$

where g is the Jacobian $|\partial x/\partial q|$ of the transformation of $\{x\}$ to $\{q\}$. The operator $(p_x^2(\partial x/\partial q)^2 F)^W$ [according to (1)] is the quantum operator corresponding to the function p_q^2F in generalized coordinates. Equation (2) may be rewritten in the form

$$(p_x^2(\partial x/\partial q)^2 F)^W = p_q^\dagger F p_q + \hbar^2 G(q) \quad (3)$$

where $p_q^\dagger$ is the adjoint of $p_q = -i\hbar\,\partial/\partial q$ as in the notation of Refs. 2 and 3. In our previous prescription[3] we suggested that the operator corresponding to p_q^2F be $p_q^\dagger F p_q$. Thus the difference in the operators (comparing our prescription with Weyl's) is a function of generalized coordinates, multiplied by $\hbar^2$. As a special case consider $F = 1$. We then find according to the *Weyl* prescription that

$$p_q^2 = p_q^\dagger p_q + \frac{\hbar^2}{2}\left[\left(\frac{1}{g}\frac{\partial g}{\partial q}\right)^2 + \frac{\partial}{\partial q}\left(\frac{1}{g}\frac{\partial g}{\partial q}\right)\right] \quad (4)$$

[1] We lose no generality by restricting ourselves to one dimension.

and according to our prescription

$$p_q^2 = p_q^\dagger p_q \quad (5)$$

For $F = g^{ik}$ such that $\frac{1}{2}p_{qi}g^{ik}p_{qk}$ becomes the classical Hamiltonian in generalized coordinates, it should be noted that in Eq. (3), $G = 0$. Hence our ordering is consistent with Weyl's concerning the generalized Hamiltonian.

3. THE OPERATOR CORRESPONDING TO THE CLASSICAL QUANTITY p_q^2: BORN-JORDAN AND SYMMETRIZED ORDERING

Using the prescription for *Born–Jordan* ordering of operators given in Reference 1, we find that

$$p_q^2 = p_q^\dagger p_q - \frac{2}{3}\hbar^2\left[\left(\frac{1}{g}\frac{\partial g}{\partial q}\right)^2 + \frac{\partial}{\partial q}\left(\frac{1}{g}\frac{\partial g}{\partial q}\right)\right] \quad (6)$$

The prescription given in Ref. 1 for *symmetrized ordering* leads to

$$p_q^2 = p_q^\dagger p_q - \hbar^2\left[\left(\frac{1}{g}\frac{\partial g}{\partial q}\right)^2 + \frac{\partial}{\partial q}\left(\frac{1}{g}\frac{\partial g}{\partial q}\right)\right]$$

In summary, from the previous sections we find that

$$p_q^2 = p_q^\dagger p_q + \alpha[R^2 + (\partial R/\partial q)], \qquad R = (1/g)\,\partial g/\partial q$$

where

$\alpha = +\hbar^2/2$	for Weyl ordering
$\alpha = -2\hbar^2/3$	for Born–Jordan ordering
$\alpha = -\hbar^2$	for symmetrized ordering
$\alpha = 0$	for our previous prescriptions

4. THE OPERATOR CORRESPONDING TO THE CLASSICAL QUANTITY p_q

By similar calculations we find that the Weyl, Born–Jordan, symmetrized, and our previous prescriptions all produce the same operator for p_q. That is,

$$\text{“}p_q\text{”} \to (p_q + p_q^\dagger)/2$$