

Reprinted from:

AJP Volume 40

### Equivalence of Momentum Operators in Generalized Coordinates

GARY R. GRUBER

Department of Physics

Hofstra University

Hempstead, L. I., New York 11550

(Received 30 May 1972)

In previous articles,<sup>1,2</sup> the author has, in effect, shown that the momentum operators in generalized coordinates can be either written as  $p_i = -i\hbar\partial/\partial q_i$ ,  $p_i^\dagger$ , or  $p_i^H$ , where  $p_i^\dagger$  denotes the adjoint of  $p_i$  and  $p_i^H$  denotes the Hermitian part of the operator  $p_i$ .  $p_i^\dagger$  was derived in Ref. (1) and (2) to be given as  $p_i^\dagger = p_i - i\hbar\partial(\ln g)/\partial q_i$ , where  $g$  is the Jacobian  $|\partial x_i/\partial q_i|$  of the transformation from Cartesian to generalized coordinates.  $p_i^H$ , which is just given as  $(p_i + p_i^\dagger)/2$  turns out to be given as  $p_i^H = p_i - \frac{1}{2}i\hbar\partial(\ln g)/\partial q_i$ . Using the above expressions for  $p_i^\dagger$  and for  $p_i^H$ , one can easily calculate the eigenfunctions and eigenvalues of the operators  $p_i$ ,  $p_i^\dagger$  and  $p_i^H$ . It is interesting to see that the eigenvalues of these three "momentum" operators are identical (and real, since the eigenvalues of  $p_i^H$  are real). See Table I.

TABLE I. Operator-eigenfunction-eigenvalue relations.

Operator	Eigenfunction	Eigenvalue
$p_i$	$\phi_N$	$\lambda_N$ (real)
$p_i^\dagger$	$\phi_N g^{-1}$	$\lambda_N$ (real)
$p_i^H$	$\phi_N g^{-1/2}$	$\lambda_N$ (real)

This shows an equivalence of these three momentum operators in generalized coordinates. It is also interesting to see that although  $p_i$  and  $p_i^\dagger$  are non-Hermitian operators, they have real eigenvalues. This is analogous to Kalnay's statement<sup>3,4</sup> that the covariant position operator in relativistic quantum mechanics is nonHermitian but nevertheless has real eigenvalues.

<sup>1</sup> G. R. Gruber, *Found. of Phys.* **1**, 3, 227 (1971).

<sup>2</sup> G. R. Gruber, *Intern. J. Theor. Phys.* **6**, 1, 31 (1972).

<sup>3</sup> A. J. Kalnay, *Phys. Rev. D* **1**, 1092 (1970).

<sup>4</sup> A. J. Kalnay, *Phys. Rev. D* **3**, 2357 (1971).